MATHCOUNTS®

2015 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Team Round problem with less that 10 sheets of scratch paper?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2015 MATHCOUNTS® Chapter Competition. Though these solutions provide creative and concise ways of solving the problems from the competition, there are certainly numerous other solutions that also lead to the correct answer, and may even be more creative or more concise! We encourage you to find numerous solutions and representations for these MATHCOUNTS problems.

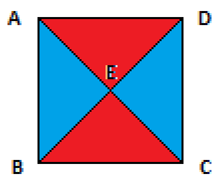
*Special thanks to volunteer author **Mady Bauer** for sharing these solutions with us and the rest of the MATHCOUNTS community!*

2015 Chapter Competition

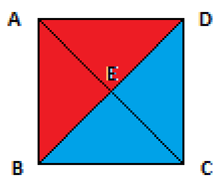
<u>Sprint Round</u>

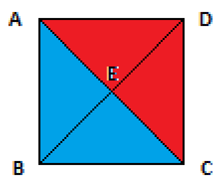
1. Given: $\frac{a}{b} = \frac{3}{5}, b = 10$
   Find: a
   $5a = 3 \times 10 = 30$
   $a = 6$ **<u>Ans.</u>**

2. Given: Square ABCD has diagonals AC and BD which intersect at E.
   Find: the number of triangles in the figure.
   There are 4 small triangles.



   There are 4 medium triangles each composed of two smaller triangles.





   4 + 4 = 8 **<u>Ans.</u>**

3. Given: Integers 1 to 100 are written.
   Find: the digit that is written the fewest number of times.
   There are ten 2-digit integers written with leading digits 1 through 9, but of course none are written with a leading 0. Because of this, 1 will be written 21 times, 2 through 9 each will be written 20 times and the digit written on 11 times is 0 **<u>Ans.</u>**

4. Given: The numbers from 1 to 8 are written in order and one of them is circled. The integers before the circled number have the same sum as the integers after the circled number.
   Find: the number that was circled.
   Write out the numbers.
   1 2 3 4 5 6 7 8
   Suppose 7 is chosen.
   1 + 2 + 3 + 4 + 5 + 6 > 8
   Suppose 6 is chosen.
   7 + 8 = 15
   1 + 2 + 3 + 4 + 5 = 15
   6 is the number chosen. 6 **<u>Ans.</u>**

5. Given: $\frac{1}{3}$ of Sally's money is spent on the bus. $\frac{1}{2}$ of the remaining money was spent at the movies leaving \$12 remaining.
   Find: how much money Sally had
   Let x = the amount of money that Sally was given. After spending money for the bus she had $\frac{2}{3}x$ left. Half of that is $\frac{1}{3}x$, which means $\frac{1}{3}x$ remains, and this amount is \$12.
   So $\frac{1}{3}x = 12$ and $x = 12 \times 3 = 36$ **<u>Ans.</u>**

6. Given: 6 added to a number results in 3 times the original number.
   Find: the original number
   Let x = the original number.
   $6 + x = 3x$
   $2x = 6$
   $x = 3$ **<u>Ans.</u>**

7. Given: Yuan gets 60% of the money for each painting that the gallery sells. Three paintings sold for \$200, \$300 and \$500.
   Find: the money that Yuan earned.
   $200 + 300 + 500 = 1000$
   $1000 \times 0.6 = 600$ **<u>Ans.</u>**

8. Given: x is 15% of 500; y is 200% of x.
   Find: $x + y$.
   $x = 0.15 \times 500 = 75$
   $y = 2 \times 75 = 150$
   $x + y = 75 + 150 = 225$ **<u>Ans.</u>**

9. Given: The four congruent triangles displayed in the square in Figure 1 are arranged to create Figure 2.
Find: the probability that a point chosen at random is in the shaded area of Figure 2.

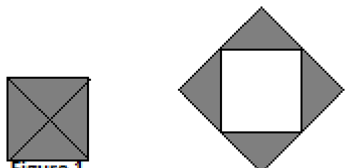


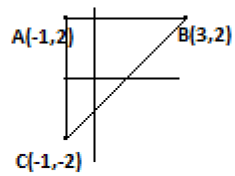
Figure 1 Figure 2

Let *x* = the length of the side of the square in Figure 1. Then the area of the smallest square is x^2.
By placing the component triangles around the space originally taken by the square, we have created a new figure that is double the area of the original figure. The area of all of Figure 2 is $2x^2$. The probability of a point being in the shaded area is $\frac{x^2}{2x^2} = \frac{1}{2}$ **Ans.**

10. Given: $\sqrt[3]{b} = 5$
Find: the value of $2b$.
$\left(\sqrt[3]{b}\right)^3 = 5^3 = 125$
$b = 125$
$2b = 125 \times 2 = 250$ **Ans.**

11. Given:
Rent: 35% of income
Car Loan: 15% of income
Utilities: 20% of income
Food: 15% of income
Savings: 10% of income
And Gasoline is $80.
Find: Martin's monthly income
35% + 15% + 20% + 15% + 10% = 95%
Therefore, everything but the Gasoline takes up 95% of Martin's income. That leaves 5% which is $80.
100% of Martin's income is
$20 \times 80 = 1600$ **Ans.**

12. Given: Points $A(-1{,}2)$ and $B(3{,}2)$ are graphed on a coordinate plane. Point C is the reflection of point A over the *x*-axis.
Find: the area of triangle ABC.



Point C is $(-1, -2)$.
The AB = $3 - (-1) = 4$
The AC = $2 - (-2) = 4$
The area of triangle ABC is
$\frac{1}{2} \times 4 \times 4 = 8$ **Ans.**

13. Find: the greatest prime factor of the product $6 \times 14 \times 22$.
$6 \times 14 \times 22 =$
$(3 \times 2) \times (7 \times 2) \times (11 \times 2)$
11 is the largest prime factor.
11 **Ans.**

14. Given: ▲ + ■ = 5; ▲ − ■ = 3
Find: ▲ + ▲.
2▲ = 5 + 3 = 8 **Ans.**

15. Given: The product of the numerator and denominator of a fraction is 60. Adding 1 to the numerator and dividing the denominator by 2 results in a fraction that is equal to 1.
Find: the fraction.
Let *n* = the numerator
Let *d* = the denominator
$n \times d = 60$
$$\frac{n+1}{\frac{d}{2}} = 1$$
$$n + 1 = \frac{d}{2}$$
$2n + 2 = d$
$n \times (2n + 2) = 60$
$2n^2 + 2n - 60 = 0$

$n^2 + n - 30 = 0$
$(n+6)(n-5) = 0$
We have *n* + 6 = 0 and *n* = − 6. But *n*/*d* is a positive common fraction so *n* must be positive. Then *n* – 5 = 0 and *n* = 5; 5*d* = 60 and *d* = 12. So $\frac{n}{d} = \frac{5}{12}$ **Ans.**

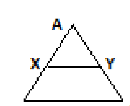
16. Given: The professor grades a paper every 10 minutes. The professor takes 20 minutes to train an assistant and then they grade 2 papers every 15 minutes.
Find: the number of papers that take the same amount of time for the professor to grade as it does for the professor to train the assistant and have both of them grade together.
Together they grade 2 papers in 15 minutes or 1 paper every 7.5 minutes.
Let *n* = the number of papers graded. Then
$10n = 20 + 7.5n$
$2.5n = 20$
$n = 8$ **Ans.**

17. Given: $b = a^2$; $c = 3b - 2$
Find: the product of all values of a for which $b = c$.
If $b = a^2$ and $c = 3b - 2$ then $c = 3a^2 - 2$
And if $b = c$ then
$a^2 = 3a^2 - 2$
$2a^2 = 2$
$a^2 = 1$
$a = \pm 1$
$1 \times -1 = -1$ **Ans.**

18. Given: A set of 6 distinct positive integers has a mean of 8, a median of 8 and no term larger than 13.
Find: the minimum possible value of any term in the set.
A mean of 8 means that the sum of the 6 values is 48.
A median of 8 means that either the third and fourth elements are 8 or the midpoint between these elements is 8, like 7 and 9, for example. In any event, the sum of the third and fourth elements are 16. Now we have 48 – 16 = 32 as the sum of the remaining four terms.
We are looking for the term of least value, so let's maximize all other values. We know that the maximum value of any term is 13. With that as the greatest term, the remaining three terms must have a sum of 32 – 13 = 19.
Since the integers are distinct, and we are using 13, let's also use 12. That leaves us with a sum of 19 – 12 = 7 for the two terms of least value.
If the sum of the first two terms is 7, they could be 1 and 6. That would give us
1, 6, 7, 9, 12, 13
1 + 6 + 7 + 9 + 12 + 13 = 48
The mean is 8, the median is 8 and the largest value is 13.
Can't get smaller than 1. **Ans.**

19. Given: $x \uparrow y = (x+y)^2$
Find: $(1 \uparrow 2) \uparrow 3$
$1 \uparrow 2 = (1+2)^2 = 3^2 = 9$
$9 \uparrow 3 = (9+3)^2 = 12^2 = 144$ **Ans.**

20. Given: Segment XY is parallel to segment BC. The area of trapezoid BCYX is 10 and the area of triangle AXY is 8.
Find: the ratio of XY to BC.



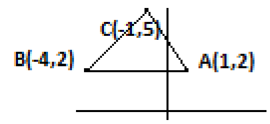
Since segments XY and BC are parallel, then $\Delta AXY \sim \Delta ABC$ (Angle-Angle). The ratio of the areas of ΔAXY to ΔABC is 8/18 = 4/9. That means the ratio of the lengths of their sides is √4/√9 and XY/BC = 2/3 **Ans.**

21. Find: the greatest absolute difference between the mean and median of 5 single-digit positive integers.
To get the smallest median we can create the set {1, 1, 1, 9, 9} – the median is 1 and the mean is $\frac{21}{5}$. The difference is $\frac{16}{5}$ **Ans.**

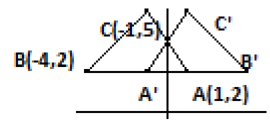
22. Given: $f\left(f(x)\right) = x^2 - 1$
Find: $f\left(f\left(f\left(f(3)\right)\right)\right)$
$f\left(f(3)\right) = 3^2 - 1 = 9 - 1 = 8$
$f\left(f(8)\right) = 8^2 - 1 = 64 - 1 = 63$ **Ans.**

23. Given: the sum of an arithmetic progression of six positive integers is 78.
Find: the greatest possible difference between consecutive terms.
Let *x* = the first term.
Let *y* = the difference between terms.
Then the terms are
$x, x + y, x + 2y, x + 3y, x + 4y, x + 5y$
$x + x + y + x + 2y + x + 3y + x + 4y +$
$x + 5y = 6x + 15y = 78$
$2x + 5y = 26$
We are looking for the largest value of *y*.
Let's start with *x* = 1.
$2 + 5y = 26$
$5y = 24$
y would not be an integer. For *y* to be an integer, the value on the right-hand side of the equation must be divisible by 5. Instead of 2 (for 2*x*) we need 6 to be the value of 2*x*, or *x* = 3.
$6 + 5y = 26$
$y = 4$ **Ans.**

24. Given: Points A, B and C are (1, 2), (−4, 2) and (−1, 5). Reflect triangle ABC across the *y*-axis.
Find: the area of the intersection of the triangle and its reflection.



Reflecting point (*x*, *y*) over the *y*-axis results in the point (−*x*, *y*). The reflection points A', B' and C' are (−1, 2), (4, 2) and (1, 5), respectively.



The line A'C' intersects line AC at their respective midpoint which is

$$\left(-1 + \frac{1 - (-1)}{2}, 2 + \frac{5 - 2}{2}\right) =$$
$$\left(-1 + 1, 2 + \frac{3}{2}\right) = \left(0, 3\frac{1}{2}\right)$$

Let's call that point D. The height of triangle A'AD is:
$h = 3\frac{1}{2} - 2 = 1\frac{1}{2}$
The base of the triangle is $1 - (-1) = 2$
The area of the triangle is:
$\frac{1}{2} \times 1\frac{1}{2} \times 2 = 1.5$ **Ans.**

25. Given: Each face of a standard die has the numbers 1 through 6. On a non-standard die, each face has an expression which is equal to 1 through 6. The expressions are
$a + 1, 2a - 5, 3a - 10, b + 8,$
$2b + 5, 3b + 10.$
Find: $a \times b$.
Clearly, the sum of all 6 expressions is
$1 + 2 + 3 + 4 + 5 + 6 = 21$
$a + 1 + 2a - 5 + 3a - 10 + b + 8 +$
$2b + 5 + 3b + 10 = 21$
$6a + 6b + 9 = 21$
$6a + 6b = 21 - 9 = 12$
$a + b = 2$
Looking at the expressions containing *b*, it is clear that b must be negative in order to get a positive value in the range of 1 to 6.

a must be positive.
Let's look at $3b + 10$. For this value to be in the range of 1 through 6, $-1 \le b \le 3$.
Suppose $b = -1$. Then
$3b + 10 = 1, 2b + 5 = 3, b + 8 = 7$
That won't work. Let's try $b = -2$.
$3b + 10 = 4, 2b + 5 = 1, b + 8 = 6$
That's good. $4 + 1 + 6 = 11$
$a + 1 + 2a - 5 + 3a - 10 =$
$21 - 11 = 10$
$6a - 14 = 10$
$6a = 24$
$a = 4$
Let's see if the other three sides will have the right values.
$a + 1 = 5, 2a - 4 = 4, 3a - 10 = 2$
That's good. So $a \times b = 4 \times -2 = -8$ **Ans.**

26. Given: $a_1 = 3, a_2 = 5, a_n = a_{n-1} - a_{n-2}$ for $n \ge 3$
Find the 2015th term.
Let's look at the first couple of terms and see if we can detect a pattern:
3, 5, 2, −3, −5, −2, 3, 5, 2, −3 etc.
The pattern repeats every 6 terms.
$$\frac{2015}{6} = 335R5$$
The fifth value of the repeating pattern is −5. **Ans.**

27. Given: A coin is flipped 8 times.
Find: the probability that equal numbers of heads and tails occur
There are
$$\frac{8!}{4!\,4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} =$$
$$\frac{7 \times 6 \times 5}{3} = 70$$
ways of getting 4 heads and 4 tails.
Therefore, the probability of flipping for 4 heads and 4 tails is
$$\frac{70}{2^8} = \frac{70}{256} = \frac{35}{128}$$ **Ans.**

28. Given: $x + |y| = y + |x|$
$-10 \le x \le 10$ and $-10 \le y \le 10$
Find: the number of ordered pairs of integers (*x*, *y*) that satisfy the equation.
If *x* and *y* are both positive we would just have $x + y = y + x$
The values of *x* and *y* that satisfy this are
$0 \le x \le 10$ and $0 \le y \le 10$
There are 11 values for *x* and 11 values for *y* for a total of $11^2 = 121$ pairs.
If *x* is negative and y is positive no pairs will work, like
$2 + |-3| = -3 + |2|$ No.
The same is true if *x* is positive and y is negative.
But, if both *x* and *y* are negative we will have some pairs that work. These pairs occur where $x = -y$
E.g., $-3 + |-3| = -3 + |-3|$
So we have 10 pairs, (−1, −1) through (−10, −10).
$121 + 10 = 131$ **Ans.**

29. Given $\frac{1}{98}$ is written a decimal.
Find: the 10th digit to the right of the decimal point.
Well using long division we can get
$$\frac{1}{98} = 0.0102040816$$
For fractions of the form 1/*n* where $10^{(m-1)} < n < 10^{m}$, it turns out that
$$1/n = \sum_{i=0}^{\infty} \frac{(10^m - n)^i}{10^{(mi+m)}}.$$
Well 98 is less than 100 = 10^2 and greater than 10 = 10^{2-1}. In our case, *n* = 98 and *m* = 2. And that crazy looking summation

formula basically means this

$$\frac{1}{98}=\frac{2^0}{10^2}+\frac{2^1}{10^4}+\frac{2^2}{10^6}+\frac{2^3}{10^8}+\frac{2^4}{10^{16}}+\cdots \text{ or}$$

.01
.0002
.000004
.00000008
.0000000016
⋮
+ ______________
.0102040816

Either way, the 10th digit is 6 **Ans.**

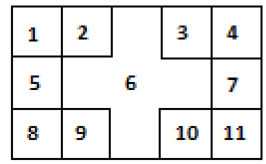
30. Given: Each exterior angle of a regular n-sided polygon is 45° larger than each exterior angle of a regular m-sided polygon. Find: the greatest possible value of m.
Each exterior angle of a regular n-sided polygon has a degree measure $360/n$, and each exterior angle of a regular m-sided polygon as a degree measure of $360/m$. We are looking for the greatest integer m such that $360/n - 45 = 360/m$. Now solve for m:
$(360 - 45n)/n = 360/m$
$360n = 360m - 45mn$
$8n = 8m - mn$
$8n = m(8 - n)$
$(8n)/(8 - n) = m$.
Now let's make a use a table of values to find the greatest integer value of m.

n	***m***	
1	8/7	
2	8/3	
3	24/5	
4	8	
5	40/3	
6	24	
7	56	
8	64/0	← Undefined
9	−72	$m < 0$ when $n \geq 9$

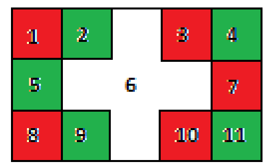
The greatest integer value is 56 **Ans.**

Target Round

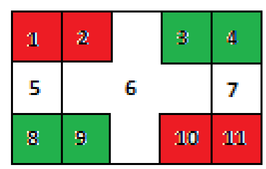
1. Find: the number of rectangles in the figure.

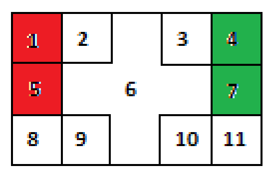


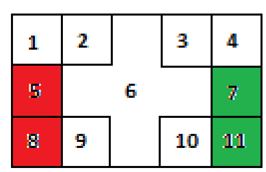
First of all, there are the small rectangles (1, 2, 3, 4, 5, 7, 8, 9, 10, 11) – 10 of them



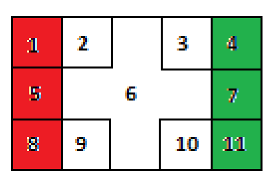
Then there are the rectangles made of two smaller rectangles (1+2, 3+4, 8+9, 10+11, 1+5, 5+8, 4+7, 7+11) – 8 of them



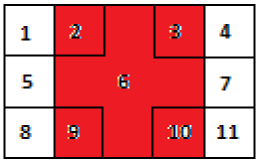




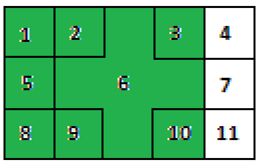
Then there are the rectangles made of three smaller rectangles (1+5+8, 4+7+11) – 2 of them.

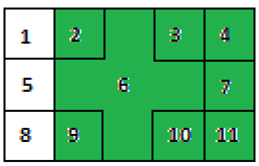


Then there is the rectangle made from everything but the two rectangles made of 3 smaller rectangles (2+3+6+9+10) – one of them.

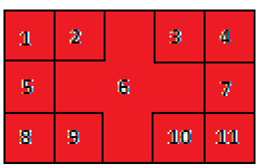


Then there are the two rectangles made from everything but one of the two rectangles made of the 3 smaller rectangles. (1+2+3+5+6+8+9+10) and (2+3+4+6+7+9+10+11).





And, finally, there is the entire rectangle.

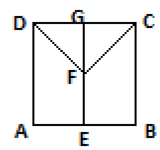


10 + 8 + 2 + 1 + 2 + 1 = 24 **Ans.**

2. Given: A batch of fudge contains 40,000 calories and fills a 9 inch x13 inch pan to a depth of 1 inch.
   Find: the number of calories in each cubic inch of fudge.
   The number of cubic inches of fudge is:
   $9 \times 13 \times 1 = 117$
   $\frac{40000}{117} \approx 341.880341 \ldots \approx 342$ **Ans.**

3. Given: The first three terms of an arithmetic sequence are $k, 2k + 3, 4k + 1$.
   Find: the fourth term.
   The adjacent terms of an arithmetic sequence differ by the same value.
   Therefore,
   $2k + 3 - k = 4k + 1 - (2k + 3)$
   $k + 3 = 4k + 1 - 2k - 3 = 2k - 2$
   $k = 5$
   The first three terms are 5, 13, 21. The difference between terms is 8, so the fourth term is $21 + 8 = 29$. **Ans.**

4. Given: Square ABCD has a side of 6. The three interior segments divide the square into two congruent trapezoids and an isosceles triangle which are all of equal area.
   Find: the length of EF.



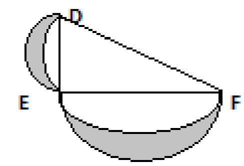
   Let EF = x.
   The area of the square is $6 \times 6 = 36$ and it is divided into three parts of equal area therefor the area of each of these is
   $36/3 = 12$
   The area of trapezoid AEFD or EBCF is:
   $\frac{1}{2}(6 + x) \times 3 = 12$
   $(6 + x) \times 3 = 24$
   $6 + x = 8$
   $x = 8 - 6 = 2$ **Ans.**

5. Find: the least positive integer that has only 1s and 0s and is a multiple of 75.
   Multiples of 75 end in 00, 25, 50 and 75. To find a value with only 1s and 0s we must consider only multiples of 75 that are divisible by 4. These multiples are:
   300, 600, 900, 1200, etc.
   Given that 1200 doesn't match we need to immediately switch to the first multiple of 75 that is 5 integers. (By the way, a multiple of 75 is also a multiple of 3 so the digits must have at least 3 ones.)
   11100 is a multiple of 75 and is the smallest

positive integer that satisfies our requirement. 11100 **Ans.**

6. Given: All points equidistant from (2, 2) and (9, 3) lie on the line $ax + by = c$.
   Find: $a + b + c$.
   One point on the line $ax + by = c$ will be the midpoint of a line drawn through the (2, 2) and (9, 3). Let (*a*, *b*) the midpoint of that line. Then:
   $a - 2 = 9 - a$
   $2a = 11$
   $a = \frac{11}{2}$
   $b - 2 = 3 - b$
   $2b = 5$
   $b = \frac{5}{2}$
   So the point is $\left(\frac{11}{2}, \frac{5}{2}\right)$.
   The line $ax + by = c$ will be perpendicular to the line between points (2, 2) and (9, 3).
   The slope between the two given points is:
   $m = \frac{3-2}{9-2} = \frac{1}{7}$
   Therefore the slope of the perpendicular line will be the negative reciprocal, -7. With the slope and one point we can now find the y intercept and write the equation of the line:
   $\frac{5}{2} = -7\left(\frac{11}{2}\right) + y_{int}$
   $y_{int} = \frac{5}{2} \times \frac{2}{11} \times \left(-\frac{1}{7}\right)$ =41
   $y = -7x + 41$ or $7x + y = 41$
   $a = 7, b = 1, c = 41$
   $a + b + c = 7 + 1 + 41 = 49$ **Ans.**

7. Given: Right triangle DEF has sides of integer length and a perimeter of 40. Semicircles with diameters DE and EF are down. Arc DEF is a semicircle.
   Find: the total area of the shaded region.



   First let's determine the sides of triangle DEF. Pythagorean triples are 3, 4, 5 (a perimeter of 12), 6, 8, 10 (a perimeter of 24), 5, 12, 13 (a perimeter of 30), 8, 15, 17 (a perimeter of 40!). And there we are.
   DE = 8, EF = 15 and DF = 17
   The area of the semicircle with diameter DE is $\frac{1}{2}\pi\left(\frac{8}{2}\right)^2 = 8\pi$
   The area of the semicircle with diameter EF is $\frac{1}{2}\pi\left(\frac{15}{2}\right)^2 = \frac{225}{8}\pi$
   The area of the semicircle with diameter DF is $\frac{1}{2}\pi\left(\frac{17}{2}\right)^2 = \frac{289}{8}\pi$
   The area of triangle DEF is
   $\frac{1}{2} \times 15 \times 8 = 60$
   The area of the shaded portion will be equal to the sum of the area of semicircle with diameter DE and the area of semicircle with diameter of EF and the area of triangle DEF minus the area of semicircle with diameter of DF.
   $8\pi + \frac{225}{8}\pi + 60 - \frac{289}{8}\pi = 60$ **Ans.**

8. Given: $f(x) + \frac{1}{x} \cdot f\left(-\frac{1}{x}\right) = 3$
   Find: $f(2)$
   Substituting $x = 2$ into the expression:
   $f(2) + \frac{1}{2} \cdot f\left(-\frac{1}{2}\right) = 3$
   $f(2) = 3 - \frac{1}{2} f\left(-\frac{1}{2}\right)$
   Substituting $x = -\frac{1}{2}$ into the expression:
   $f\left(-\frac{1}{2}\right) + \frac{1}{-\frac{1}{2}} f\left(-\frac{1}{-\frac{1}{2}}\right) = 3$
   $f\left(-\frac{1}{2}\right) - 2f(2) = 3$

$f\left(-\frac{1}{2}\right) = 3 + 2f(2)$

Now, substitute the second equation into the first:

$f(2) = 3 - \frac{1}{2}f\left(-\frac{1}{2}\right) = 3 - \frac{1}{2}(3 + 2f(2))$

$2f(2) = 6 - 3 - 2f(2)$

$4f(2) = 3$

$f(2) = \frac{3}{4}$ **Ans.**

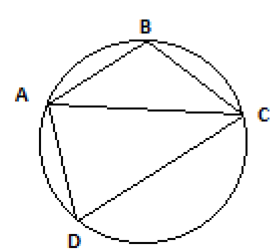
Team Round

1. Given: Devin cleaned his room on Jan. 1 and went to the gym. He goes to the gym every other day and cleans his room every Thursday.
   Find: how many days he cleans his room and goes to the gym as well.
   Jan 1 was a Thursday. Therefore, he goes to the gym on Thursday, Saturday, Monday, Wednesday, Friday, Sunday, Tuesday, Thursday – i.e., every two weeks he'll be cleaning his room and going to the gym. So, how many Thursdays do we have in 2015?
   There are 365 days in 2015.
   $\frac{365}{7} = 52R1$
   Therefore we actually have 53 Thursdays in the year. Starting with the first Thursday, there are actually 27 Thursdays where Devin will clean his room and go to the gym.
   27 **Ans.**

2. Given: The sum of the areas of two equilateral triangles is equal to twice the area of a third equilateral triangle. The side length of the first triangle is 3 and the side length of the second triangle is 5.
   Find: the side length of the third triangle.
   The formula for the area of an equilateral triangle is
   $A = \frac{\sqrt{3}}{4}s^2$ where s is the side of the triangle.
   The area of the first triangle is
   $\frac{\sqrt{3}}{4}(3)^2 = \frac{9}{4}\sqrt{3}$
   The area of the second triangle is:
   $\frac{\sqrt{3}}{4}(5)^2 = \frac{25}{4}\sqrt{3}$
   The sum of the area of both triangles is:
   $\frac{9}{4}\sqrt{3} + \frac{25}{4}\sqrt{3} = \frac{34}{4}\sqrt{3} = \frac{17}{2}\sqrt{3}$
   This is twice the area of the third triangle, so the area of the third triangle is:
   $\frac{17}{4}\sqrt{3}$
   If s is the side of the third triangle:
   $\frac{\sqrt{3}}{4}s^2 = \frac{17}{4}\sqrt{3}$
   $s^2 = 17$
   $s = \sqrt{17}$ **Ans.**

3. Given: The cheetah's speed is 75 mph. The sloth's speed is 13 feet per minute.
   Find: how many times faster the cheetah's speed is.
   75 mph is
   $\frac{75 \times 5280}{60} = \frac{5 \times 5280}{4} = 5 \times 1320 =$
   6600 feet per minute.
   $\frac{6600}{13} \approx 507.692307 \ldots \approx 508$ times **Ans.**

4. Given: AB || CD and the measure of $m\angle$ADC = 50°. $m\angle$BAC = $m\angle$BCA
   Find: $m\angle$BAD



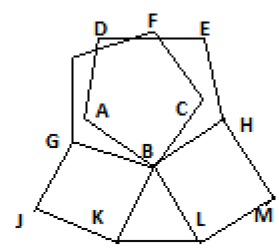
Let x = the measure of $\angle$BAC.

$m\angle$BCA = x and $m\angle$ABC = 180 – 2x
Since $AB||CD$, $m\angle$ACD = $m\angle$BAC = x
Thus, $m\angle$BCD = $m\angle$BCA + $m\angle$ACD = 2x
$m\angle$CAD) = 180 − $m\angle$ADC − $m\angle$ACD = 180 – 50 – x = 130 − x
$m\angle$BAD = $m\angle$BAC + $m\angle$CAD = 130 **Ans.**

5. Given: $\frac{1}{x} + \frac{1}{y} = 2$
$\frac{1}{xy} = 4$
Find: $\frac{x+y}{2}$
$\frac{1}{4} = xy$
$\frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{xy} = \frac{x+y}{xy} = 2$
$x + y = 2xy$
$\frac{x+y}{2} = xy = \frac{1}{4}$ **Ans.**

6. Find: the greatest possible sum of the digits of a six-digit number that is a multiple of 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.
Find the least common multiple of 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11
$2 \times 3 \times 2 \times 5 \times 7 \times 2 \times 3 \times 11 = 27720$
The largest 6-digit multiple of 27720 is 997920 (27720×36). (Note that any multiple of 27720 will end in 0.) 99792 is not bad. The sum of the digits is 34.
The smallest 6-digit multiple of 27720 is 110880 (27720×4). The sum of the digits is 18. Probably better to look at more of the larger multiples.
$27720 \times 35 = 970200$ - way too many zeroes! Note that multiplying 27720 by any multiple of 5 will result in at least two of the digits being 0, so let's remove 5, 10, 15, 20, 25, 30 and 35.
Multiplying 27720 by 32 or 34 gives 877040 and 942480. Sums are 26 and 27, respectively.
Multiplying 27720 by 26 or 28 gives 720720 and 776160. Sums are 18 and 27, respectively.
Multiplying 27720 by 22 and 24 gives 609840 and 665280. Sums are 27 and 27, respectively.
Multiplying 22720 by 16 and 18 gives 363520 and 498960. Sums are 19 and 36, respectively. Aha! 498960 beats 977920. Continuing on.
Multiplying 22720 by 12 and 14 gives 332640 and 318080. Sums are 18 and 20, respectively.
Multiplying 22720 by 6 and 8 gives 136320 and 181760. Sums are 15 and 23, respectively.
36 **Ans.**

7. Given: A side of each square coincides with a side of the equilateral triangle. One side of each regular pentagon coincides with a side of one of the squares.
Find: the degree measure of $\angle$ABC



Triangle KBL is an equilateral triangle. This makes the measure of $\angle$KBL = 60°.
$\angle$GBK and $\angle$LBH and are both angles within squares. Therefore, each is 90°.
The sum of all three angles is 240°. This makes the measure of $\angle$GBH = 120°.
$\angle$GBC is an angle of the pentagon FCBGD; its measure is 108°. $\angle$ABH is an angle of pentagon ABHED; its measure is 108°.
$m\angle$GBC = 108 = $m\angle$GBA + $m\angle$ABC
$m\angle$ABH = 108 = $m\angle$ABC + $m\angle$CBH

$m\angle$GBH = 120 = $m\angle$GBA + $m\angle$ABC + $m\angle$CBH

120 = 108 + $m\angle$CBH

12 = $m\angle$CBH

120 = $m\angle$GBA + 108

12 = $m\angle$GBA

120 = 12 + $m\angle$ABC + 12

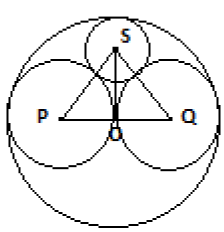
$m\angle$ABC = 120 – 24 = 96 **Ans.**

8. Given: A stairstep number is a number whose digits are strictly increasing in value from left to right.
   Find: How many positive integers with two or more digits are stairstep numbers.
   There are several things to note:
   - 0 cannot be part of a stairstep number. We don't start numbers with a leading zero (even if we did, it wouldn't matter) and inserting a zero anywhere else in the number violates the concept of a stairstep number.
   - The largest stairstep number is 9 digits long because if you have 10 digits, one of them will have to be equal to or less than an adjacent number.
   Let's start with 9 digit numbers.
   Obviously there is only 1: 123456789.
   Now let's look at 8 digit numbers. They are: 12345678, 12345679, 12345689, 12345789, 12346789, 12356789, 12456789, 13456789, 23456789. Note that each number has a single digit removed from 123456789. And this is the way to solve the problem. We are just looking for ways of picking a number of digits to remove from 123456789.
   For a 7 digit number remove 2 of the digits.
   $\frac{9!}{7!\,2!} = \frac{9 \times 8}{2} = 36$
   For a 6 digit number, remove 3 of the digits.
   $\frac{9!}{6!\,3!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 3 \times 4 \times 7 = 84$
   For a 5 digit number, remove 4 of the digits.
   $\frac{9!}{5!\,4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 3 \times 7 \times 6 = 126$
   For a 4 digit number, remove 5 of the digits.
   $\frac{9!}{4!\,5!} = \frac{9!}{5!\,4!} = 126$
   For a 3 digit number, remove 6 of the digits.
   $\frac{9!}{3!\,6!} = \frac{9!}{6!\,3!} = 84$
   And for a 2 digit number, remove 7 of the digits.
   $\frac{9!}{2!\,7!} = \frac{9!}{7!\,2!} = 36$
   Therefore, the total number of stairstep numbers is:
   $1 + 9 + 36 + 84 + 126 + 126 + 84$
   $+ 36 = 502$ **Ans.**

9. Given: We move either on the *x* or *y* axis (but not both at the same time). The first step moves 1 unit, the second 2 units, etc.
   Find: the number of paths from (0, 0) to (14, 14).
   How many ways can we add several numbers that sum up to 14?
   We can't go more than 4 digits to get to 14. Why? 1 + 2 + 3 + 4 = 10 and 1 + 2 + 3 + 4 + 5 = 15 – so if we just did consecutive moves on one axis, 5 moves would be too far.
   Let's look at 4 moves.
   1 + 2 + 3 + 8
   1 + 2 + 4 + 7
   1 + 2 + 5 + 6
   1 + 3 + 4 + 7
   1 + 3 + 5 + 6
   2 + 3 + 4 + 5
   And that's it for 4 moves.
   Now 3 moves.
   1 + 2 + 11
   1 + 3 + 10
   1 + 4 + 9
   1 + 5 + 8
   1 + 6 + 7

2 + 3 + 9
2 + 4 + 8
2 + 5 + 7
3 + 4 + 7
Whichever set we pick for the *x*-axis, we have to pick another set that uses integers other than the first set we've picked and the union of both sets of integers give us a set of consecutive integers starting with 1.
By inspection:
One combination can be 1 + 2 + 5 + 6 and 3 + 4 + 7
We can switch the 1 and 2 from the set of 4 integers with the 3 from the set of 3 integers. The second combination is
3 + 5 + 6 and 1 + 2 + 4 + 7
Switch 6 with 2 + 4. The third combination:
2 + 3 + 4 + 5 and 1 + 6 + 7
Switch 3 + 4 with 7. The fourth combination:
2 + 5 + 7 and 1 + 3 + 4 + 6.
8 **<u>Ans.</u>**

10. Given: Circles P and Q are tangent to circle O and tangent to each other at point O.
Circle S is tangent to circles O, P and Q. PQ = 24 cm.
Find: the diameter of circle S.
Create the triangle SPQ by connecting points S, P and Q. Draw a perpendicular from S to O, the height of the triangle.



Since PQ = 24, 24 is the radius of circle O.
The radius of circles P and Q are each 12.
Let *x* = the radius of circle S.
Then OS = 24 – *x*.
PO = OQ = 12
The length of SP or SQ is 12 + *x*.
$(24 - x)^2 + 12^2 = (12 + x)^2$
$24^2 - 48x + x^2 + 12^2 =$
$12^2 + 24x + x^2$
$24^2 - 48x = 24x$
$24 - 2x = x$
$24 = 3x$
$x = 8$
The diameter is twice that, or 16. **<u>Ans.</u>**