MATHCOUNTS®

2024 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide detailed solutions to the Sprint, Target and Team Rounds of the 2024 MATHCOUNTS Chapter Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

Sprint 1

Arrange the terms in increasing order: 529.12, 529.13, 529.14, 529.14, 529.15, 529.16. The median is the middle value. There are 6 terms numbered 1 through 6, so the middle term is $\#(1+6)/2 = 3.5$. Therefore, we need the average of terms 3 and 4, which are the same: **529.14**.

Sprint 2

The two lines meet at the crossing of vertical grid line $x = 2$ and the horizontal grid line $y = 3$, so the (x, y)-coordinate is $(\mathbf{2}, \mathbf{3})$.

Sprint 3

$-3^2 = -(3^2) = -9$; $(-3)^2 = 9$. Therefore, $|-3^2 - (-3)^2| = |-9-9| = |-18| = \mathbf{18}$.

Sprint 4

$f(4) = 2^4 + 5 = 16 + 5 = \mathbf{21}$.

Sprint 5

The perimeter p of a square with sides of length s is $p = 4s$; the area of the square is $A = s^2$. We are given $A = 9\text{ cm}^2$, so $s^2 = 9\text{ cm}^2$ and $s = 3$ cm. Therefore, $p = 4 \times 3\text{ cm} = \mathbf{12}$ cm.

Sprint 6

$3b + 12 + 7 = 5b + 11 \rightarrow 12 + 7 - 11 = 5b - 3b \rightarrow 8 = 2b$. Therefore, $b = \mathbf{4}$.

Sprint 7

$p = 17s \rightarrow s = p/17 = 306\text{ cm}/17 = \mathbf{18}$ cm.

Sprint 8

$\#(\text{meat}) \times \#(\text{veg}) = 3 \times 5 = \mathbf{15}$.

Sprint 9

$(5\sqrt{2})^2 = 5^2 \times 2 = 25 \times 2 = \mathbf{50}$.

Sprint 10

$30\,\cancel{\text{s}} \times \frac{1\text{ min}}{60\,\cancel{\text{s}}} = \frac{30}{60}\text{ min} = \frac{1}{2}\text{ min} = 0.5\text{ min}$, so 2 min 30 s = 2.5 min. This means he will type 2.5 times 84 words, which is 84 + 84 + 42 = **210** words.

Sprint 11

$750 = n_{\text{now}} = (1 + 25\%)n_{\text{last}} = \frac{5}{4}n_{\text{last}}$, so $n_{\text{last}} = \frac{4}{5}(750) = 4(150) = \mathbf{600}$.

Sprint 12

$12\,\cancel{\text{b}} \times \frac{3\,\cancel{\text{s}}}{5\,\cancel{\text{b}}} \times \frac{\cancel{5}\text{ z}}{4\,\cancel{\text{s}}} = \frac{12\times 3\text{ z}}{4} = 3 \times 3\text{ z} = \mathbf{9}$ z.

Sprint 13

As the angles of a triangle, $180 = x + (3x - 10) + (x - 10) = 5x - 20$, so $x = (180 + 20)/5 = 40$. Therefore, the angles are 40°, 110° and 30°, the largest of which is **110**°.

Sprint 14

$\frac{15\,\cancel{\text{min}}}{2\,\cancel{\text{br}}} \times 42\,\cancel{\text{br}} \times \frac{1\text{ h}}{60\,\cancel{\text{min}}} = \frac{15\times 42}{60\times 2}\text{ h} = \frac{1}{4} \times 21\text{ h} = \mathbf{5\frac{1}{4}}$ h.

Sprint 15

$l = 1.5w$. Thus, $24\text{ in}^2 = A = lw = 1.5w^2 \rightarrow 1.5w^2 = 24\text{ in}^2 \rightarrow \frac{3}{2}w^2 = 24\text{ in}^2 \rightarrow w^2 = (24\text{ in}^2)(\frac{2}{3}) \rightarrow$ $w^2 = 16\text{ in}^2 \rightarrow \sqrt{w^2} = \sqrt{16\text{ in}^2} \rightarrow w = 4\text{ in} \rightarrow l = 1.5 \times 4\text{ in} = 6\text{ in}$. Therefore, $p = 2(l + w) =$ $2(6\text{ in} + 4\text{ in}) = 2(10\text{ in}) =$ **20** in.

Sprint 16

When the product of two real numbers is fixed, the sum and the absolute difference are maximized when the two values are separated as much as possible. The smaller one of the numbers is, the greater the other is to have a fixed product, widening the separation. Because the values are restricted to integers, the least possible positive integer is 1, in which case the other value is 36, with a greatest difference of $36 - 1 =$ **35**. [Allowing negative values yields the same result.]

Sprint 17

To have at most five positive factors, a positive integer must be: (1) the fourth or less power of a prime or (2) the product of two distinct primes. The greatest prime less than 100 is 97, which has 2 factors. Both 98 and 99 are the product of one prime (2 and 11, respectively) and the square of another prime (7 and 3, respectively), thus having 6 factors. Therefore, the answer is **97**.

Sprint 18

The area of a right triangle with legs of length a and b is $ab/2$, so $24\text{ cm}^2 = (8\text{ cm})b/2$, so $b = 2 \times 24\text{ cm}^2/(8\text{ cm}) = 6\text{ cm}$. The legs of the larger triangle are, therefore, $8\text{ cm} + 2\text{ cm} = 10\text{ cm}$ and $6\text{ cm} + 2\text{ cm} = 8\text{ cm}$, making the area $(10\text{ cm})(8\text{ cm})/2 =$ **40** cm^2.

Sprint 19

Let c_{bef} and c_{aft} be the number of candies before and after the meeting, respectively, and m be the number of members. $c_{\text{aft}} = 27$; $c_{\text{bef}} = 2c_{\text{aft}} = 54$. Therefore, $c_{\text{bef}} + 17 - 2m - (m - 1) = c_{\text{aft}} \rightarrow$ $54 + 17 + 1 - 3m = 27 \rightarrow 3m = 72 - 27 = 45 \rightarrow m = 45/3 =$ **15**.

Sprint 20

$$\frac{1}{1+\frac{2}{1+\frac{3}{1+\frac{5}{1+7}}}} = \frac{1}{1+\frac{2}{1+\frac{3}{1+\frac{5}{8}}}} = \frac{1}{1+\frac{2}{1+\frac{3}{(\frac{13}{8})}}} = \frac{1}{1+\frac{2}{1+\frac{24}{13}}} = \frac{1}{1+\frac{2}{(\frac{37}{13})}} = \frac{1}{1+\frac{26}{37}}$$

$$= \frac{1}{(\frac{63}{37})} = \mathbf{\frac{37}{63}}.$$

Sprint 21

$42\,000 = 5(k) \times 5(k+1) \times 5(k+2) = 125(k)(k+1)(k+2)$, so

$$k(k+1)(k+2) = \frac{42\,000}{125} = \frac{8 \times 42\,\cancel{000}}{\cancel{1\,000}} = 8 \times 42 = 8 \times 7 \times 6$$

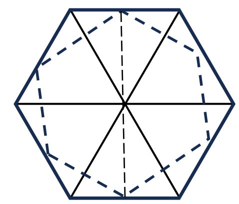
Therefore, $6 \times 5 = 30$, $7 \times 5 = 35$, $8 \times 5 = 40$ are the three multiples of 5, of which **30** is the least.

Sprint 22

Let n be an even number (for us, $n = 2022$). Two arithmetic series are under consideration:
$1 + 2 + \ldots + (n-1) + n$, which has n terms averaging $(1+n)/2$, so a sum of $A = n(n+1)/2$.
$2 + 4 + \ldots + (n-2) + n$, which has $n/2$ terms averaging $(2+n)/2$, so a sum of $B = n(n+2)/4$.
Thus, $A - B = [(n^2+n)/2 - (n^2+2n)/4] = [(2n^2+2n) - (n^2+2n)]/4 = n^2/4 = (n/2)^2$. For

$n = 2022$, we want $(1011)^2 \equiv 11^2 \pmod{1000}$, and $11^2 = \mathbf{121}$. Alternatively, $1011^2 = 1{,}022{,}121$, which is 1,022.121 when divided by 1000, or 1022 with a remainder of **121**.

Sprint 23



The larger hexagon (hexagon A) can be partitioned into 6 congruent equilateral triangles, each having side length s. We see that the distance from one vertex of the hexagon to the opposite vertex (segment PQ) is $2s$. The ratio of the height to the base of an equilateral triangle is always $\sqrt{3}/2$, so the distance from the center of the hexagons to point M is $\sqrt{3}s/2$, and the distance from the midpoint of one side of hexagon A to the midpoint of the opposite side (segment MN) is $\sqrt{3}s$. Thus, for dotted hexagon B, the vertex-to-opposite-vertex distance (also segment MN) is $\sqrt{3}s$. Therefore, the linear scaling (diagonal compared to diagonal) of B to A is $\sqrt{3}s/2s = \sqrt{3}/2$. With the area scaling being proportional to the square of the linear scaling, the result is an area ratio of hexagon B to hexagon A equal to $\left(\sqrt{3}/2\right)^2 = \mathbf{3/4}$.

Sprint 24

Sprint: 23, 24, 26, 28, 28, 29; Target: 10, 10, 12, 12, 14. 16. The highest score must be at least 39 because S29 must go with at least 10 on Target. However, in that scenario, there remains only one T10, so one of the S28s must go with at least T12, for a total of 40 that contradicts the goal of 39. If we continue, one S28 with T10 for 38, the other S28 with one T12 for 40, S26 with the other T12 for 38, S24 with T14 for 38, and S23 with T16 for 39. We have constructed a scenario where **40** is the top score and occurs only once.

Sprint 25

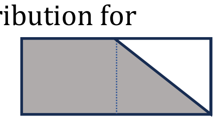
$(6t - 6)° + (11t - 1)° + [(180° - (16t + 1)°] = 180°$. Therefore, $(t - 8)° = 0$, so $t = \mathbf{8}$.

Sprint 26

First, we factor each of the trinomials in the original equation: $x^2 + x - 2 = (x + 2)(x - 1)$; $x^2 - 5x + 6 = (x - 2)(x - 3)$; $x^2 - 4x + 3 = (x - 1)(x - 3)$. Now, rewriting the original equation, we have $(x - 1)(x + 2)(x - 1)(x - 2)(x - 3) = 12(x - 1)(x - 3)(x + 2) \rightarrow$ $[(x + 2)(x - 1)(x - 3)](x - 1)(x - 2) = 12[(x + 2)(x - 1)(x - 3)]$, $\rightarrow$ bringing everything to the right side, factoring out $[(x + 2)(x - 1)(x - 3)]$ from the resulting two terms and multiplying out $(x - 1)(x - 2)$ results in $0 = [(x + 2)(x - 1)(x - 3)][(x^2 - 3x + 2) - 12] =$ $[(x + 2)(x - 1)(x - 3)](x^2 - 3x - 10) \rightarrow (x + 2)(x - 1)(x - 3)\,(x + 2)\,(x - 5) = 0 \rightarrow (x + 2)^2(x - 1)(x - 3)\,(x - 5) = 0$, which has solutions -2 [double], 1, 3, 5. The sum of the positive solutions is $1 + 3 + 5 = \mathbf{9}$.

Sprint 27

Leaving before 3:15 guarantees an early arrival. Leaving at time t after 3:15, for $0 \text{ min} \le t \le 15 \text{ min}$, allows $45 \text{ min} - t$ to arrive on time, which changes linearly from guaranteed to make it at $t = 0$ min to guaranteed to not make it at $t = 15$ min. Thus, the shape of the probability distribution for Edward to arrive early as a function of departure time is given by the graph:



Thus, we are missing a triangular portion that is half the width of the rectangle and the same height, so the triangle has $\frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{4}$ the area of the rectangle, yielding a probability of $\frac{1}{4}$ being late, so $1 - \frac{1}{4} = \mathbf{\frac{3}{4}}$ of being early.

Sprint 28

Divisibility by 99 means divisibility by 9 and by 11. Divisibility by 9 is not an issue because in base ten, an integer is divisible by 9 if and only if the sum of its digits is divisible by 9; we have each digit 1 through 9 exactly once, always summing to 45, a multiple of 9, regardless of digit order. Let the specified number be represented by the decimal representation RsT,uVw,XyZ, where each letter represents a digit of so-far unknown value in the range 1 through 9, with distinct letters for distinct digits. Divisibility of our number by 11 means the difference of the sum of the digits in the odd positions (our capital letters) and the sum of the digits in the even positions (our lower-case letters) is divisible by 11 (so the difference is 0, ± 11, ± 22, ± 33, etc.). We'll let A = (R + T + V + X + Z) and B = (s + u + w + y), and we've already acknowledged A + B = 45. For A – B = 0 to be true, we would need A = B = 22.5, so A – B cannot be 0. For A – B = ± 11, we could have one value be 17 and the other be 28…something to work with! Hold that thought. [Note that A – B = ± 22 only works if A and B are both even or both odd, which can't be true with A + B = 45. For the possibility of A – B = ± 33, we get one value of 6 and one of 39…another option we need to consider.] Let's see if we can create 17 and 28 for A and B (not knowing which would be 17 or 28). There are 5 digits in odd positions and 4 digits in even positions. Since we are looking for the smallest possible integer, we could ideally start:

A → 1 + 3 + 5 + __ + __ and B → 2 + 4 + __ + __ (which would be integer 123,45__, __ __ __)

Starting here, can we get B to 17 or 28? We would need the remaining two entries to add to 11 or 22, which is impossible with the digits we have left. So, this start is too hopeful. Let's back out a bit and try:

A → 1 + 3 + __ + __ + __ and B → 2 + 4 + __ + __ (which would be integer 123,4 __ __, __ __ __)

Again, we need the remaining two entries of B to add to 11 or 22, which is now possible with 5 and 6! Thus, we can use:

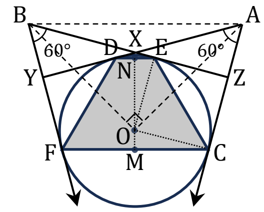
A → 1 + 3 + __ + __ + __ and B → 2 + 4 + 5 + 6 (which would be integer 123,4__5,__6__) and putting in the remaining digits in a way that yields the smallest integer gives us **123,475,869**.

Note, we have just found the answer assuming our A and B were the sums of 17 and 28. We said earlier we would need to check A and B being the sums of 6 and 39, too. However, it is impossible for 4 or 5 digits to add to 6, so there are no options for A or B to equal 6.

Sprint 29

It is easier to deal the multiples of 8, instead of the non-multiples of 8, but we need to subtract our result from 1 to get the desired answer. There are three basic cases we need to handle: *Case 1*. All three dice show multiples of 2, so we have at least 3 factors of 2 to yield the desired 8. For each die, probability is 1/2 for even, so combined yields $(1/2)^3 = 1/8$. *Case 2*. Two dice show 4 and one shows odd for a multiple of 16, which is a multiple of 8. The two 4s each have probability 1/6 and odd has probability 1/2, but any of the 3 dice can be the odd, so combined yields $(1/6)^2(1/2)(3) = 1/24$. *Case 3*. One die shows 4, one die shows 2 or 6, and one die shows odd, with probability 1/6, 1/3 and 1/2, respectively, and there are $3! = 6$ arrangements of which die shows which number, so combined yields $(1/6)(1/3)(1/2)(6) = 1/6$. Therefore, we need $1 - (1/8 + 1/24 + 1/6) =$
$1 - 8/24 = 16/24 =$ **2/3**.

Sprint 30

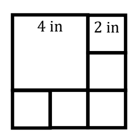


Let M be the midpoint of $\overline{FC}$, N be the midpoint of $\overline{DE}$, X be the intersection of $\overrightarrow{AD}$ and $\overrightarrow{BE}$, Y be the intersection of $\overrightarrow{AD}$ and $\overrightarrow{BF}$, and Z be the intersection of $\overrightarrow{BE}$ and $\overrightarrow{AC}$. Circle O has radius $r = 6$ cm. Triangle ABO is right isosceles, with angles ABO and BAO having measure 45°. Segment OB bisects angle EBF and segment OA bisects angle DAC so angles OBE and OAD have measure 30°. Thus, angles ABX and BAX have measure $45° - 30° = 15°$ and angle BXA has measure $180° - 2(30°) = 150°$; vertical angle DXE likewise has measure 150°. OD ⊥ DA and OE ⊥ EB, so angle DOE has measure $360° - (150° + 2(90°)) = 30°$. Segments $\overline{DE}$ and $\overline{FC}$ are the bases of isosceles triangles DOE and FOC, respectively, as well as being the bases for trapezoid CEDF in question. Segment $\overline{MN}$ bisects the two bases and their respective triangles, resulting in four congruent 15°-75°-90° right triangles FOM, COM, ODN and OEN with hypotenuse of length r. The 15°-75°-90° right triangle does not show up very often in MATHCOUNTS, but often enough that it merits memorization by the extra diligent students, to go with the 30°-60°-90° and 45°-45°-90° right triangles: $\sqrt{6} - \sqrt{2} : \sqrt{6} + \sqrt{2} : 4$. Segments DN and EN are the shorter leg, while FM and CM are the longer leg. Thus, $DE = 2\frac{\sqrt{6}-\sqrt{2}}{4}r = 3(\sqrt{6} - \sqrt{2})$ cm and $FC = 2\frac{\sqrt{6}+\sqrt{2}}{4}r = 3(\sqrt{6} + \sqrt{2})$ cm. The average of the two base lengths is $3\sqrt{6}$ cm. The height of the trapezoid is MO + ON [a shorter leg and a longer leg, thus the average base length], $3\sqrt{6}$ cm. Therefore, the shaded area is the product of these two values, $3\sqrt{6}$ cm × $3\sqrt{6}$ cm = **54** cm².

Target 1

$\frac{130}{8} = 16\frac{1}{4}$, to which we must apply the ceiling operator [the least integer greater than or equal to the operand] to round up to the integer **17**, because occupying a fraction of a table (a 17th table) means the whole table is needed. [NOTE: The ceiling of x is expressed symbolically as $\lceil x \rceil$, which is used also in the Target 6 solution.]

Target 2



$1 \times (4 \times 4)\ \text{in}^2 + 5 \times (2 \times 2)\ \text{in}^2 = 16\ \text{in}^2 + 20\ \text{in}^2 = \mathbf{36}\ \text{in}^2$.

Target 3

$t_{\text{rest}} = t_{\text{total}} - t_{\text{moving}} = \frac{1200\ \text{mi}}{48\ \text{mi/h}} - \frac{1200\ \text{mi}}{60\ \text{mi/h}} = 25\ \text{h} - 20\ \text{h} = 5\ \text{h} \times \frac{60\ \text{min}}{1\ \text{h}} = \mathbf{300}\ \text{min}$.

Target 4

$BE = \sqrt{325^2 - 323^2}\ \text{cm} = \sqrt{648 \times 2}\ \text{cm} = \sqrt{1296}\ \text{cm} = 36\ \text{cm}$; $EC = 204\ \text{cm} - 36\ \text{cm} = 168\ \text{cm}$.
$DF = \sqrt{325^2 - 204^2}\ \text{cm} = \sqrt{105\,625 - 41\,616}\ \text{cm} = \sqrt{64\,009}\ \text{cm} = 253\ \text{cm}$.
$FC = 323\ \text{cm} - 253\ \text{cm} = 70\ \text{cm}$. $EF = \sqrt{168^2 + 70^2}\ \text{cm} = 7\sqrt{24^2 + 10^2}\ \text{cm} = 7(26\ \text{cm}) = \mathbf{182}\ \text{cm}$.

Target 5

$3x^2 - 27 = (3x - 27)^2 = 9x^2 - 162x + 729 \rightarrow 0 = 6x^2 - 162x + 756 = 6(x^2 - 27x + 126) \rightarrow$
$0 = x^2 - 27x + 126 = (x - 6)(x - 21)$, so **21** is the greatest solution.

Target 6
$V = \$1n_1 + \$9n_9 + \$81n_{81} + \$729n_{729}$, for nonnegative integers $n_1, n_9, n_{81}, n_{729}$ with sum 500.
$V = \$1n_1 + \$9n_9 + \$81n_{81} + \$729n_{729} = \$1n_1 + \$(1+8)n_9 + \$(1+80)n_{81} + \$(1+728)n_{729}$
$= \$1(n_1 + n_9 + n_{81} + n_{729}) + \$8n_9 + \$8(10n_{81}) + \$8(91n_{729}) = \$500 + \$8(n_9 + 10n_{81} + 91n_{729})$,
which has the form $\$500 + \$8k$ for some nonnegative integer k. The least value of k for which $\$500 + \$8k > \$10\,000$ is $\lceil (10\,000 - 500)/8 \rceil = 1188$, for which $V = \$500 + \$8(1188) = \$10\,004$. But can we actually achieve \$10 004? Yes, if we have at most 13 bills of \$729 and the other 487 bills are all \$1, the total is \$9964. Swapping 5 of those \$1 bills for \$9 bills yields \$**10 004**. [NOTE: $\lceil x \rceil$ denotes the ceiling of x and is described in the Target 1 solution.]

Target 7
We start with 9 mL of solution of which $0.5 \times 9\text{mL} = 4.5$ mL is syrup. Add water to the solution until 4.5 mL of syrup is 30 % (that is 0.3) of the solution, so the total solution will be $4.5 \text{ mL}/0.3 =$ **15** mL.

Target 8
Let's see if we can find a path where every possible length of hop is used. The first hop must be the longest, and the longest possible hop is one corner to the opposite corner (length $\sqrt{4^2+4^2} = \sqrt{32}$). Then back adjacent to first pad (length $\sqrt{4^2+3^2} = \sqrt{25}$). Then to opposite side adjacent to pad 2 (length $\sqrt{4^2+2^2} = \sqrt{20}$). Then length $\sqrt{3^2+3^2} = \sqrt{18}$, then length $\sqrt{4^2+1^2} = \sqrt{17}$, then to opposite side (length $\sqrt{4^2+0^2} = \sqrt{16}$), then length $\sqrt{3^2+2^2} = \sqrt{13}$, then length $\sqrt{3^2+1^2} = \sqrt{10}$, then length $\sqrt{3^2+0^2} = \sqrt{9}$, then length $\sqrt{2^2+2^2} = \sqrt{8}$, then length $\sqrt{2^2+1^2} = \sqrt{5}$, then length $\sqrt{2^2+0^2} = \sqrt{4}$, then length $\sqrt{1^2+1^2} = \sqrt{2}$, then length $\sqrt{1^2+0^2} = \sqrt{1}$, for a total of **15** pads. One such path is shown in the figure. [Note that we want the count of occupied lily pads, not the count of hops, which is 1 too few at 14. Each small square in the figure represents a lily pad, and the number in the small square, if any, gives the sequence of visiting that lily pad.]

1	5	7		
3		12	10	
8	11	13		4
15	14	6	9	2

Team 1

Four darts cannot achieve more than $4 \times 10 = 40$ points; $45 = 5 \times 9$ works, so **5** darts.

Team 2
Adding a to each data value also increases the mean by a, so we'll find the mean of the original values and then add 5: $\frac{40+38+13+10+7}{5} + 5 = \frac{108}{5} + 5 = 21.6 + 5 = \mathbf{26.6}$.

Team 3
$S = 2(\sqrt{5} \times 3\sqrt{2}\text{ cm}^2 + \sqrt{5} \times \sqrt{2}\text{ cm}^2 + 3\sqrt{2} \times \sqrt{2}\text{ cm}^2) = 2(3\sqrt{10}\text{ cm}^2 + \sqrt{10}\text{ cm}^2 + 12\text{ cm}^2) =$
$(8\sqrt{10} + 12)\text{ cm}^2 = 37.298\ldots\text{ cm}^2$, which rounded to the nearest tenth of a square centimeter is **37.3** cm^2.

Team 4
$\frac{2\text{ def}}{5\ \cancel{\text{bx}\times 8\text{ cmp/bx}}} \times \left(25\ \cancel{\text{cr}} \times \frac{27\ \cancel{\text{bx}}}{1\ \cancel{\text{cr}}} \times \frac{\cancel{8\text{ cmp}}}{1\ \cancel{\text{bx}}}\right) = \mathbf{270}$ defective computers.

Team 5

(1): $R + A = 20$; (2): $3R - 5A = 20$. Multiply equation (1) by 5 and add to equation (2) to cancel A:

$5R + 5A = 100$

$\underline{3R - 5A = \ \ 20}$

$8R \qquad = 120$, so $R = 120/8 =$ **15**.

Team 6

$S(2023)$ is the sum of the first 2023 terms; $S(2024)$ is the sum of the first 2024 terms. Thus, term 2024 is the difference between $S(2024)$ and $S(2023)$ and can be expressed as:

$S(2024) - S(2023) = [3(2024^2) - 2(2024)] - [3(2023^2) - 2(2023)] =$

$3(2024^2) - 2(2024) - 3(2023^2) + 2(2023) =$

$3(2024^2 - 2023^2) - 2(2024 - 2023) = 3(2024^2 - 2023^2) - 2 =$

$3(2024 + 2023)(2024 - 2023) - 2 = 3(4047) - 2 =$ **12,139**.

Team 7

For $a = 1$, b can be any integer from 1 through 11, with $c = 12 - b$. That is 11 triples; except for $a = b = 1$, we can swap the a and b components to obtain 10 additional triples, so 21 triples so far. For $a = 2$, b can be any integer from 2 through 5 [the case of (2, 1, 9) having already been taken care of with the swapping of a and b in handling the a = 1 cases with b = 2], with $c = 12 - 2b$, for 4 more triples (25 total so far); except for $a = b = 2$, we can swap the a and b components to obtain 3 additional triples, so 28 triples so far. For $b \geq a \geq 3$, there is only one more triple that satisfies all criteria, (3, 3, 3), so we have a total of **29** distinct qualifying triples.

Team 8

All of the following triangles, listed in order of increasing size, are similar: DAE, ABD, CDE, CBA. The ratio of area for DAE to ABD is 14:18, thus 7:9, so, the linear scaling ratio of DAE to ABD is $\sqrt{7} : 3$. Thus, the hypotenuse length ratio for triangles DAE to ABD is $\overline{DA} : \overline{AB} :: \sqrt{7} : 3$. Likewise, the ratio of the length of the longest legs for the same two triangles is $\overline{ED} : \overline{DA} :: \sqrt{7} : 3$. Combining these two ratios yields $\overline{ED} : \overline{AB} :: 7 : 9$. Now, ED and AB are the short legs of triangles CDE and CBA, respectively. Thus, the areas of CDE and CBA are in the ratio $7^2 : 9^2$, that is, 49 : 81. Now, the difference in area of those two triangles is the area of trapezoid ABDE, which we are given to be $18\text{ cm}^2 + 14\text{ cm}^2 = 32\text{ cm}^2$. The ratio of trapezoid ABDE area to triangle CDE are is $81 - 49 : 49$, thus, 32 : 49. Therefore, the area of triangle CDE is $\cancel{32}\text{ cm}^2 \times 49/\cancel{32} =$ **49** cm^2.

Team 9

If we cut two end links (1 & 2, 1 & 7 or 6 & 7 for 3 options), 1 segment remains. If we cut one end link and one interior link, or two adjacent interior links (1 & 3, 1 & 4, 1 & 5, 1 & 6, 2 & 3, 2 & 7, 3 & 4, 3 & 7, 4 & 5, 4 & 7, 5 & 6 or 5 & 7 for 12 options), 2 segments remain. If we cut two non-adjacent interior links (2 & 4, 2 & 5, 2 & 6, 3 & 5, 3 & 6 or 4 & 6 for 6 options), 3 segments remain. Therefore, the expected value for the count of remaining segments is $(3 \times 1 + 12 \times 2 + 6 \times 3)/(3 + 12 + 6)$ $= 45/21 = 15/7 = \mathbf{2\frac{1}{7}}$.

Team 10

Because 2024 is a multiple of the three spacings (8, 11, 23) involved (in fact, $2024 = 8 \times 11 \times 23$) and the numbers are relatively prime, the probabilities of ending up with any of the various coins are mutually independent, so we can approach this as a probability problem! The probability of any one coin being a quarter is $\frac{1}{23}$, so there are $\frac{1}{23} \times 2024 = 8 \times 11 = 88$ quarters. Every 11th coin is a dime except where replaced by a quarter, so the probability of a coin (1) being a dime and (2) not being a quarter is $\frac{1}{11} \times \frac{22}{23} = \frac{2}{23}$, so $\frac{2}{23} \times 2024 = 2 \times 88 = 176$ dimes. Every 8th coin is a nickel unless replaced by a dime or quarter, so the probability of a coin (1) being a nickel, (2) not being a dime and (3) not being a quarter is $\frac{1}{8} \times \frac{10}{11} \times \frac{22}{23} = \frac{5}{46}$, so $\frac{5}{46} \times 2024 = \frac{5}{2} \times 88 = 220$ nickels. The remaining $(2024 - 88 - 176 - 220 = 1540)$ coins are pennies. The total value is $1540¢ + 220 \times 5¢ + 176 \times 10¢ + 88 \times 25¢ = 1540¢ + 1100¢ + 1760¢ + 2200¢ = \mathbf{6600}¢$.